A solution to the problem posed by Byland and Scialom

M. Yu. Zotov
D. V. Skobeltsyn Institute of Nuclear Physics,
Moscow State University, Moscow 119899, Russia
(e-mail: `zotov@eas.sinp.msu.ru`)

Recently, Byland and Scialom studied the evolution of the Bianchi I, the Bianchi III and the Kantowski–Sachs universe on the basis of dynamical systems methods (*Phys. Rev.* **D57**, 6065 (1998), gr-qc/9802043). In particular, they have pointed out a problem to determine the stability properties of one of the degenerate critical points of the corresponding dynamical system. Here we give a solution, showing that this point is unstable both to the past and to the future. We also discuss the asymptotic behavior of the trajectories in the vicinity of another critical point.

In one of their recent works, Byland and Scialom studied the evolution of the Bianchi I, the Bianchi III and the Kantowski–Sachs universe in a model with a real scalar field and a convex positive potential [1]. A considerable part of the investigation was devoted to the analysis of the asymptotic behavior and the stability properties of the solutions of the Einstein–Klein–Gordon equations

$$\begin{aligned}
\dot\theta &= -\frac{1}{3}\,\theta^2 - 2\sigma^2 + V(\varphi) - \psi^2,\\
\dot\sigma &= -\frac{1}{3\sqrt{3}}\,\theta^2 - \theta\sigma + \frac{1}{\sqrt{3}}\left(\sigma^2 + V(\varphi) + \frac{1}{2}\,\psi^2\right),\\
\dot\varphi &= \psi,\\
\dot\psi &= -\theta\psi - \frac{dV}{d\varphi},
\end{aligned}$$

where θ is the function of the expansion rate, $\sigma_{\mu\nu}$ is the shear tensor of the hypersurface of constant time, $\sigma = \frac{1}{2}\,\sigma_{\mu\nu}\sigma^{\mu\nu}$, φ is the scalar field, and V is a convex positive potential; an overdot stands for derivatives with respect to t (see Eqs. (8)–(11) in [1]).

The analysis was based on determining the stability properties of the critical points of the dynamical system

$$\begin{aligned}
S' &= -\frac{1}{3\sqrt{3}} - \frac{2}{3}S + \frac{1}{2\sqrt{3}}(2S^2 + 2U^2 + P^2) + FS,\\
U' &= \left(\frac{1}{3} - \frac{\lambda}{2}P + F\right)U, \qquad\qquad (1)\\
P' &= -\frac{2}{3}P + \lambda U^2 + FP,
\end{aligned}$$

where $S = \sigma/\theta$, $U = \sqrt{V}/\theta$, $P = \psi/\theta$, $F = 2S^2 - U^2 + P^2$, λ is given by $V = V_0\, e^{-\lambda\phi}$, and a prime stands for derivatives with respect to τ defined by $d\tau = \theta\, dt$ (see Eqs. (24)–(26) in [1]). Here we omit an equation for θ, since Eqs. (1) do not contain this function.

It was found in [1] that the dynamical system (1) has the following critical points:

$$P_1: \quad S = -\frac{1}{2\sqrt{3}},\ U = 0,\ P = 0$$

$$P_2: \quad S = 0,\ U = \sqrt{\frac{6-\lambda^2}{18}},\ P = \frac{\lambda}{3}, \quad \lambda \le \sqrt{6}$$

$$P_3: \quad S = \frac{1}{2\sqrt{3}}\frac{2-\lambda^2}{1+\lambda^2},\ U = \frac{\sqrt{2+\lambda^2}}{\sqrt{2}\,(1+\lambda^2)},\ P = \frac{\lambda}{1+\lambda^2}$$

$$\Sigma: \quad U = 0,\ 3S^2 + \frac{3}{2}P^2 = 1, \quad S \in [-1/\sqrt{3}, 1/\sqrt{3}].$$

In particular, it was shown that the point P_2 has the eigenvalues $\varepsilon_1 = -1+\lambda^2/6$ (twice) and $\varepsilon_2 = -2/3+\lambda^2/3$. Thus, it is stable for $\lambda < \sqrt{2}$ and unstable for $\sqrt{2} < \lambda < \sqrt{6}$. It was also found that P_2 is also unstable for $\lambda = \sqrt{2}$. The problem is to determine the stability properties of P_2 for the case $\lambda = \sqrt{6}$, in which P_2 is degenerate with the eigenvalues $\varepsilon_1 = 0$ (twice) and $\varepsilon_2 = 4/3$. Here we shall demonstrate that the point P_2 is unstable (both to the future and to the past).

Notice that for the case $\lambda = \sqrt{6}$, P_2 belongs to the ellipsis Σ. Thus, one of the zero eigenvalues corresponds to the fact that Σ is a one-dimensional critical set. In order to prove that the critical point P_2 is unstable both to the past and to the future, it is sufficient to show that there is a projection of this point, which is unstable. To do this, we rewrite the dynamical system (1) as

$$\begin{aligned}
S' &= 2\alpha\left(\frac{1}{2\sqrt{3}} + S\right)\bar P + \frac{1}{2\sqrt{3}}(2S^2 + 2U^2 + \bar P^2) + \bar F S,\\
U' &= \left(\frac{\alpha}{2}\,\bar P + \bar F\right)U, \qquad\qquad (2)\\
\bar P' &= \frac{4}{3}\bar P + \alpha\,(3U^2 + 2\bar P^2) + (\alpha + \bar P)\,\bar F,
\end{aligned}$$

where $\bar P = P - \alpha$, $\alpha = \sqrt{2/3}$, and $\bar F = 2S^2 - U^2 + \bar P^2$. The point P_2 now corresponds to the origin, $(S, U, \bar P) = (0, 0, 0)$.

Consider the projection of (2) in the plane $S = 0$. The equations for U and $\bar P$ may now be written as

$$U' = \left(\frac{\alpha}{2}\,\bar P - U^2 + \bar P^2\right)U, \qquad (3a)$$

$$\bar P' = \frac{4}{3}\bar P + (2\alpha - \bar P)\,U^2 + (3\alpha + \bar P)\,\bar P^2. \qquad (3b)$$

Denote the right hand sides of Eqs. (3a) and (3b) as $\mathcal{U}(U,\bar P)$ and $\mathcal{P}(U,\bar P)$ respectively. The idea is to find a solution $\bar P = f(U)$ of the equation $\mathcal{P}(U,\bar P) = 0$ in a neighborhood of $U = 0$, to substitute it into $\mathcal{U}(U,\bar P)$:

$$\mathcal{U}(U, f(U)) = a_m U^m + \dots,$$

and then to examine whether the power m of the leading term is even or odd and, in the latter case, to check the sign of a_m (see, e.g., [2]).

It is easy to see that the corresponding solution in our case assumes the form

$$\bar{P} = f(U) = -\sqrt{\frac{3}{2}}\, U^2 + O(U^4).$$

Therefore,

$$\mathcal{U}(U, f(U)) = -\frac{3}{2}\, U^3 + O(U^7).$$

Thus, $m = 3$ is odd, and a_m is negative. It follows immediately from the standard results for the two-dimensional dynamical systems that the point $(U, \bar{P}) = (0, 0)$ is a topological saddle. Hence, it is unstable both to the future and to the past.

Let us mention that the same conclusion may be obtained basing on the results of the center manifold theory (see, e.g., [3]).

We remark that P_2 is not the only degenerate point of Σ. Recall that the eigenvalues of Σ are [1]

$$\varepsilon_1 = 0, \quad \varepsilon_2 = 1 - \frac{\lambda}{2}\, P, \quad \varepsilon_3 = \frac{2}{3}(2 + \sqrt{3}\, S).$$

Thus, for any point of Σ, except for the points $(P, S) = (0, \pm 1/\sqrt{3})$, there exists $\lambda = 2/P$, $\lambda \in [-\sqrt{6}, \sqrt{6}]$ such that ε_2 turns to zero, thus making this point degenerate. The stability properties of these degenerate points may be studied in a way similar to the above.

Finally, let us make some comments on the behavior of the trajectories of (1) in the vicinity of the point P_1. It was found in [1] that this point has the eigenvalues $\varepsilon_{S,P} = -1/2$ and $\varepsilon_U = 1/2$. It was also claimed that starting around P_1, the critical point can never be reached by any solutions of the dynamical system. This is not quite correct. It follows from the standard results of the dynamical systems theory (see, e.g., [4], Theorem 6.1) that in a neighborhood of P_1 there exists a two-dimensional stable manifold W^s and a one-dimensional unstable manifold W^u. The trajectories lying on these manifolds tend to P_1 as $\tau \to \infty$ and $\tau \to -\infty$ respectively. One can easily find the asymptotic behavior of these solutions.

Namely, let us introduce $\bar{S} = S + 1/(2\sqrt{3})$. Then the dynamical system (1) reads as

$$\begin{aligned}
\bar{S}' &= -\frac{1}{2}\bar{S} - \frac{1}{2\sqrt{3}}\left(4\bar{S}^2 - 3U^2\right) + \bar{F}\bar{S}, \\
U' &= \frac{1}{2}\left(1 - \lambda P - \frac{4}{\sqrt{3}}\,\bar{S} + 2\bar{F}\right)U, \qquad (4) \\
P' &= -\frac{1}{2}P - \frac{2}{\sqrt{3}}\,\bar{S}P + \lambda U^2 + \bar{F}P,
\end{aligned}$$

where $\bar{F} = 2\bar{S}^2 - U^2 + P^2$. The point P_1 now corresponds to the origin, $(\bar{S}, U, P) = (0, 0, 0)$. The eigenvectors of this point are $\zeta_{\bar{S}} = (1, 0, 0)$, $\zeta_P = (0, 0, 1)$, and $\zeta_U = (0, 1, 0)$. Now one can see that the trajectories on W^s take the form

$$U \equiv 0, \quad S \approx -\frac{1}{2\sqrt{3}} + C_S\, e^{-\tau/2}, \quad P \approx C_P\, e^{-\tau/2},$$

as $\tau \to \infty$, where C_S and C_P are arbitrary constants, $C_S^2 + C_P^2 \neq 0$. (One can also obtain these solutions in a parametric and partially in an explicit form (for $\bar{S} \equiv 0$).) Notice that S here is just a linear function of P. We also mention that one of the trajectories on W^s, namely,

$$U \equiv 0, \quad S = -\frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{2}}\, P \quad \text{for } P \in \,]0, \sqrt{2/3}[,$$

joins P_1 to P_2 if $\lambda = \sqrt{6}$.

In order to obtain the asymptotic behavior of the trajectories on W^u, we notice that for these trajectories $\bar{S} = o(U)$ and $P = o(U)$ as $U \to 0$. This allows us to consider only the leading terms in (4):

$$\bar{S}' = -\frac{1}{2}\,\bar{S} + \frac{3}{2}\,U^2, \quad U' = \frac{1}{2}\,U, \quad P' = -\frac{1}{2}\,P + \lambda U^2.$$

It follows immediately that the outgoing trajectories take the form

$$U \approx C_U\, e^{\tau/2}, \quad S \approx -\frac{1}{2\sqrt{3}} + \frac{1}{\sqrt{3}}\,U^2, \quad P \approx \frac{2}{3}\,\lambda U^2,$$

as $\tau \to -\infty$, where C_U is a nonzero constant.

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